No-slip motion of a spherical magnet on top of a conductive plate

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The trajectory of a spherical magnet which rolls without slipping on a conductive plate is modelled. A time stepping $\vec{T}-\Omega$ method is used to find the electromagnetic force and torque. The trajectory is computed for different initial conditions and compared to ini- -tial conditions and compared to performed experiments.

Index terms—Magnets, conducting materials, eddy currents, finite element methods.

I. Introduction

THE COUPLING between the partial differential equations (PDEs) that describe the electromagnetic field and the ordinary differential equations (ODEs) that determine the motion of a moving conducting object can be a particularly difficult problem especially when the control parameters of the ODEs use variables that issue from the field problems.

Most coupled mechanical-electromagnetic problems concern devices which have only one mechanical degree of freedom [1]-[2]-[3], amongst the exceptions are magneto-hydrodynamical flows. Most cases describe either the field-circuite coupling [4] or the FE mechanical coupling [5].

If a moving spherical magnet is on top of a conductive plate, eddy currents are induced and tend to slow down the spherical magnet until it stops. This is a transient problem, the position and the direction of the magnetization are the five mechanical degrees of freedom. The control parameters of the ODEs are the Lorentz force and the torque, which are calculated by solving the EM field problem. Similarly, the electromagnetic field depends on the position of the moving magnet. The aim here is to find the dynamics of the magnet under different initial conditions.

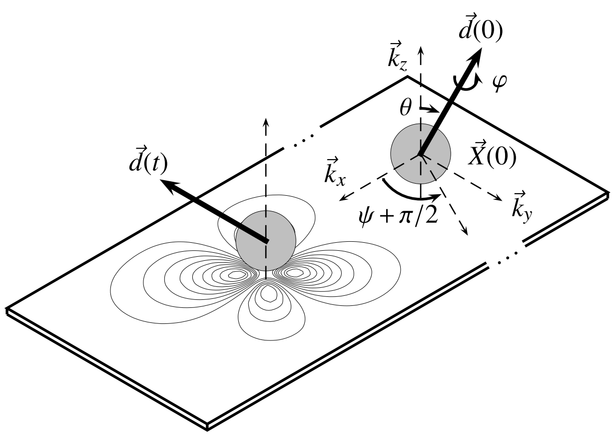

Fig. 1. Sketch of the magnet on top of a conducting plate,the eddy-currents are also shown on the figure.

II. Governing equations

The spherical magnet (radius R) is assumed to roll without slipping on a horizontal plane. The position $\vec{X} = X(t)\,\vec{k}_x + Y(t)\,\vec{k}_y + R\,\vec{k}_z$ of its center of gravity is obviously time-dependent, as is the (planar) velocity $\dot{\vec{X}} = \dot{X}\,\vec{k}_x + \dot{Y}\,\vec{k}_y$. The magnetization is directed along

$$\vec{d}(t) = -\sin\theta\sin\psi\,\vec{k}_x + \sin\theta\cos\psi\,\vec{k}_y + \cos\theta\,\vec{k}_z \tag{1}$$

where ψ, θ are the precession and nutation Euler angles (Fig. 1). The intrinsic rotation φ around the axis of direction $\vec{d}$ has to be considered to describe the motion of the magnet but has no influence whatsoever on the magnetic field.

Due to the no-slip condition, the instantaneous rotation vector $\vec{\omega}$ can be expressed as a function of the velocity components. It has only one kinematically free component ω_z:

$$\vec{\omega}(t) = -\dot{Y}(t)/R\,\vec{k}_x + \dot{X}(t)/R\,\vec{k}_y + \omega_z(t)\,\vec{k}_z \tag{2}$$

The motion of the magnet is governed by the following dynamical system of equations:

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \\ \dot{X} \\ \dot{Y} \\ \omega_z \\ \theta \\ \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \frac{5}{7m}(f_x + \frac{\Gamma_y}{R}) \\ \frac{5}{7m}(f_y - \frac{\Gamma_x}{R}) \\ \frac{5}{2mR^2}\Gamma_z \\ \frac{1}{R}(\dot{X}\sin\psi - \dot{Y}\cos\psi) \\ \omega_z + \frac{1}{R\tan\theta}(\dot{X}\cos\psi + \dot{Y}\sin\psi) \\ -\frac{1}{R\sin\theta}(\dot{X}\cos\psi + \dot{Y}\sin\psi) \end{pmatrix} \tag{3}$$

where m is the mass of the magnet ; and $f_x\,\vec{k}_x + f_y\,\vec{k}_y$, $\Gamma_x\,\vec{k}_x + \Gamma_y\,\vec{k}_y + \Gamma_z\,\vec{k}_z$ are the applied force and the torque respectively.

If the plane is the top of a copper plate, the force and torque are both due to eddy currents induced by the motion of the magnet (rolling friction is neglected).

The eddy-currents have to be determined, a time-dependent $\vec{T}-\Omega$ formulation is used here. The magnetic field due to the magnet is given by the dipolar expression (outside the magnet):

$$\vec{h}_s = \vec{\nabla}_{\vec{x}}(\Omega_s) \text{ with } \Omega_s = -\frac{\mathcal{M}\,\vec{d}\cdot x\vec{X}}{4\pi\,|x\vec{X}|^3} \text{ and } x\vec{X} = \vec{x}-\vec{X} \quad (4)$$

where $\mathcal{M}$ is the magnetic moment of the magnet. The time derivative of $\vec{h}_s$ is then $\dot{\vec{h}}_s = \vec{\nabla}_{\vec{x}}\left(\dot{\Omega}_s\right)$ where

$$\dot{\Omega}_s = \frac{\mathcal{M}}{4\pi\,|x\vec{X}|^3}\,\vec{d}\cdot\left(\dot{\vec{X}} - 3\frac{\dot{\vec{X}}\cdot x\vec{X}}{|x\vec{X}|^2}\,x\vec{X} - \vec{\omega}\times x\vec{X}\right) \quad (5)$$

Notice that the source electromotive force depends on the components of the dynamical system (3).

In this $\vec{T}-\Omega$ formulation, the system of PDEs to be solved both in the conductive plate D and outside the plate $E_3 - D$ is:

$$\vec{\nabla}\times\left(\frac{1}{\sigma}\vec{\nabla}\times\vec{T}\right) + \mu_0\,\partial_t\left(\vec{T}+\vec{\nabla}\Omega\right) + \mu_0\,\dot{\vec{h}}_s = \vec{0} \quad \text{in } D \quad (6)$$

$$\vec{\nabla}\cdot\vec{T} = \vec{0} \quad \text{in } D \quad (7)$$

$$\vec{T}\times\vec{n} = \vec{0} \quad \text{on } \partial D \quad ; \quad \vec{T} = 0 \quad \text{in } E_3 - D \quad (8)$$

$$\vec{\nabla}\cdot\left[\mu_0\left(\vec{\nabla}\Omega + \vec{T}\right)\right] = 0 \quad \text{in } E_3 \quad (9)$$

The eddy current density and magnetic field are $\vec{j} = \vec{\nabla}\times\vec{T}$ and $\vec{h} = \vec{T} + \vec{\nabla}\Omega$ and the total Lorentz force and torque exerted on the plate are respectively ($d\vec{x}^3 = dx\,dy\,dz$)

$$\begin{aligned} -\vec{f} &= \int_D \left(\vec{\nabla}\times\vec{T}\right)\times\mu_0\vec{h}_s\,d\vec{x}^3 \\ -\vec{\Gamma} &= \int_D x\vec{X}\times\left(\left(\vec{\nabla}\times\vec{T}\right)\times\mu_0\vec{h}_s\right)\,d\vec{x}^3 \end{aligned} \quad (10)$$

The force and torque are exactly equal and opposite to those exerted on the magnet.

The system (6-9) has to be solved in time. A time step τ is given ; $\vec{T}$, Ω and an estimation (described below) of $\dot{\vec{h}}_s$ at discrete times $n\tau$ are denoted $\vec{T}^n$, Ω^n and $\dot{\vec{h}}_s^{*\,n}$. The full time-implicit weak form version of the $\vec{T}$-system (6-7-8) is: $\forall\vec{T}'$ such that $\vec{T}'\times\vec{n} = \vec{0}$ on ∂D, $\vec{\nabla}\cdot\vec{T}' = 0$ on D

$$\int_D\left(\frac{1}{\sigma}\vec{\nabla}\times\vec{T}^{n+1}\cdot\vec{\nabla}\times\vec{T}' + \frac{\mu_0}{\tau}\left[\vec{T}^{n+1}+\vec{\nabla}\Omega^{n+1}\right]\cdot\vec{T}'\right)d\vec{x}^3 = \quad (11)$$

$$\int_D \frac{\mu_0}{\tau}\left[\vec{T}^n + \vec{\nabla}\Omega^n\right]\cdot\vec{T}'d\vec{x}^3 - \int_D \mu_0\,\dot{\vec{h}}_s^{*\,n+1}\cdot\vec{T}'d\vec{x}^3$$

and, since the only contribution of $\vec{\nabla}\cdot\vec{T}$ is the jump of $\vec{T}\cdot\vec{n}$ on ∂D, the Biot and Savart form of (9) reduces to:

$$\Omega^{n+1}(\vec{x}) = \frac{\mu_0}{4\pi}\int_{\partial D}\frac{\vec{T}^n(\vec{y})\cdot\vec{n}}{|\vec{x}-\vec{y}|}d\vec{y}^2 \quad (12)$$

If $\dot{\vec{h}}_s^{*\,n+1}$ were known, (11-12) would allow the computation of ($\vec{T}^{n+1}$, Ω^{n+1}) from ($\vec{T}^n$, Ω^n) (at the cost of iterations between (11) and (12) to avoid the direct inversion of a non-sparse matrix). But if $\dot{\vec{h}}_s^{*\,n+1}$ corresponds to $\dot{\vec{h}}_s^{n+1}$, the value of $\dot{\vec{h}}$ at the time $(n+1)\tau$, the components of the dynamical system (3) have to be computed implicitly. It is possible to solve such a system implicitly, but it requires an iterative process to handle the non-linearities. To overcome this drawback, a kind of semi-implicit method is used. ($\vec{X}^{n+1}, \vec{d}^{n+1}, \vec{\omega}^{n+1}$) are found with a time step of (3), with an explicit high order Runge-Kutta method (Dormand-Prince 5/4), where the components of force and torque (10) are frozen at time $n\tau$. Then (11-12) are solved implicitly with $\dot{\vec{h}}_s^{*\,n+1} = \dot{\vec{h}}_s(\vec{X}^{n+1}, \vec{d}^{n+1}, \vec{\omega}^{n+1})$.

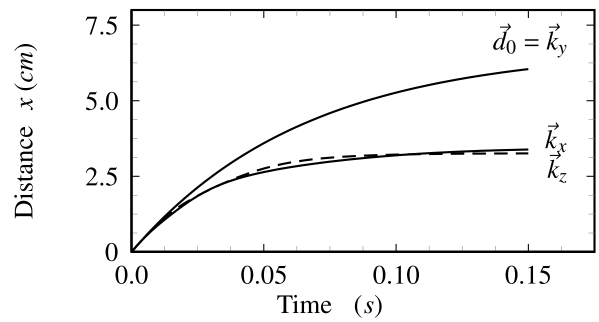

Fig. 2. Distance travelled vs time for various initial directions of magnetization ($\vec{d}_0 = \vec{d}(t=0)$) being respectively equal to $\vec{k}_x, \vec{k}_y, \vec{k}_z$).

III. Results

As a first test, to evaluate the accuracy of the model, a spherical Nd-Fe-B magnet ($R = 6\,mm$, $\mathcal{M} = 1\,A\,m^2$) is thrown with an initial speed $\dot{\vec{X}}(t=0) = \dot{x}_0\,\vec{k}_x$ ($\dot{x}_0 = 1\,m/s$) on a copper plate (thickness $5\,mm$). The initial direction of magnetization varies. The spherical magnet has a linear trajectory $\vec{X}(t) = x(t)\vec{k}_x$. Fig. 2 shows the time dependence of the distance x when $\vec{d}_0 = \vec{k}_x, \vec{k}_y, \vec{k}_z$. For any initial direction, the distance travelled before the sphere comes to a stop is between the one for $\vec{k}_x, \vec{k}_z$ (most efficient braking situation) and the one for $\vec{k}_y$ (least efficient case). It is difficult to determine the actual direction of $\vec{d}_0$ in the experiments (the magnet comes from a ramp with an elbow). However many experiments confirmed that the total distance travelled before coming to a full stop is between 3 and $6\,cm$.

IV. References

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